Group Distributionally Robust Optimization

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Outline

- Introduction
- Related Work
- Stochastic Approximation of GDRO
  - Stochastic Mirror Descent
  - Non-oblivious Online Learning
- GDRO with Imbalanced Data
  - Stochastic Mirror Descent with Non-uniform Sampling
  - Stochastic Mirror-Prox Algorithm with Mini-batches
- Conclusion

Statistical Machine Learning

- Risk Minimization

$$\min_{\mathbf{w}\in\mathcal{W}} \left\{ R_0(\mathbf{w}) = \mathrm{E}_{\mathbf{z}\sim\mathcal{P}_0}\left[\ell(\mathbf{w};\mathbf{z})\right]\right\}$$

  - $\mathbf{w}$ denotes the learning model, $\mathbf{z}$ is a random sample
  - $\mathcal{P}_0$ is a unknown distribution, $\ell(\cdot;\cdot)$ is a loss function

- Examples

  - SVM
$$\min_{\mathbf{w}\in\mathcal{W}} \mathrm{E}_{(\mathbf{x},y)\sim\mathcal{P}_0}\left[\max(1-y\mathbf{w}^\top\mathbf{x},0)\right] + \frac{\lambda}{2}\|\mathbf{w}\|^2$$
  - Linear Regression
$$\min_{\mathbf{w}\in\mathcal{W}} \mathrm{E}_{(\mathbf{x},y)\sim\mathcal{P}_0}\left[(y-\mathbf{w}^\top x)^2\right]$$

Optimization Approaches I

I. Sample Average Approximation (SAA)

I. Empirical Risk Minimization (ERM)

$$\min_{\mathbf{w}\in\mathcal{W}} \frac{1}{n}\sum_{i=1}^{n}\ell(\mathbf{w};\mathbf{z}_i)$$

- $\mathbf{z}_1, \ldots, \mathbf{z}_n$ are independently sampled from $\mathcal{P}_0$

□ Deterministic Optimization

✓ Gradient Descent, Mirror Descent, Newton's method

□ Stochastic Optimization

✓ Stochastic Gradient Descent, Stochastic Mirror Descent

✓ Variance Reduction (Johnson and Zhang, 2013; Zhang et al., 2013)

Optimization Approaches II

II. Stochastic Approximation (SA)

$$\min_{\mathbf{w} \in \mathcal{W}} \left\{ R_0(\mathbf{w}) = \mathrm{E}_{\mathbf{z} \sim \mathcal{P}_0} \left[\ell(\mathbf{w}; \mathbf{z}) \right] \right\}$$

- Stochastic Gradient Descent (SGD)

$$\mathbf{w}_{t+1} = \Pi_{\mathcal{W}} \left[\mathbf{w}_t - \eta \nabla \ell(\mathbf{w}_t; \mathbf{z}_t) \right], \quad \mathbf{z}_t \sim \mathcal{P}_0$$

✓ The stochastic gradient is unbiased

$$\mathrm{E}\left[\nabla \ell(\mathbf{w}_t; \mathbf{z}_t) \right] = \nabla R_0(\mathbf{w}_t)$$

At least in theory, we cannot reuse samples!

Optimization Approaches II

II. Stochastic Approximation (SA)

$$\min_{\mathbf{w}\in\mathcal{W}} \left\{R_0(\mathbf{w}) = \mathrm{E}_{\mathbf{z}\sim\mathcal{P}_0}\left[\ell(\mathbf{w};\mathbf{z})\right]\right\}$$

- Stochastic Gradient Descent (SGD)

$$\mathbf{w}_{t+1} = \Pi_{\mathcal{W}}\left[\mathbf{w}_t - \eta\nabla\ell(\mathbf{w}_t, \mathbf{z}_t)\right], \quad \mathbf{z}_t \sim \mathcal{P}_0$$

- Stochastic Mirror Descent (SMD) (Nemirovski et al., 2009)

$$\mathbf{w}_{t+1} = \operatorname*{argmin}_{\mathbf{w}\in\mathcal{W}} \left\{\eta\langle\nabla\ell(\mathbf{w}_t;\mathbf{z}_t), \mathbf{w}-\mathbf{w}_t\rangle + B(\mathbf{w},\mathbf{w}_t)\right\}$$

$$B(\mathbf{u},\mathbf{v}) = \nu(\mathbf{u}) - \left[\nu(\mathbf{v}) + \langle\nabla\nu(\mathbf{v}), \mathbf{u}-\mathbf{v}\rangle\right]$$

✓ SMD becomes SGD when $\nu(\mathbf{w}) = \|\mathbf{w}\|^2/2$

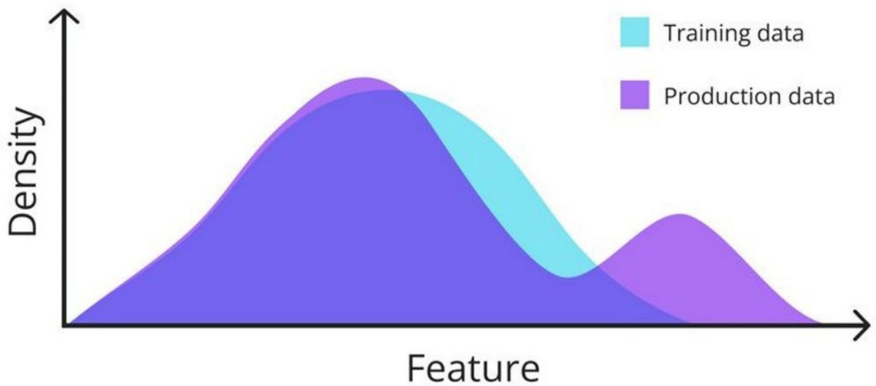
Statistical Machine Learning

- Theoretical Guarantee
  - SAA and SA

$$\underbrace{R_0(\bar{\mathbf{w}}) - \min_{\mathbf{w}\in\mathcal{W}} R_0(\mathbf{w})}_{\text{Excess Risk}} = O\left(\frac{1}{\sqrt{n}}\right), \quad O\left(\frac{1}{n}\right)$$

- Limitations

Lack robustness when distribution shifts



https://www.nannyml.com/blog/6-ways-to-address-data-distribution-shift

Distributionally Robust Optimization (DRO)

➢ Formulation of DRO

$$\min_{\mathbf{w}\in\mathcal{W}} \sup_{\mathcal{P}\in\mathcal{S}(\mathcal{P}_0)} \left\{ \mathrm{E}_{\mathbf{z}\sim\mathcal{P}} \left[\ell(\mathbf{w};\mathbf{z}) \right] \right\}$$

- $\mathcal{S}(\mathcal{P}_0)$ denotes a set of probability distributions around $\mathcal{P}_0$

➢ A Vast Amount of Literature

- Robust optimization (Scarf, 1958; Ben-Tal et al., 2009)
- Asymptotic properties (Duchi and Namkoong, 2021)
- Constructions of the neighborhood (Delage and Ye, 2010; Ben-Tal et al., 2013; Esfahani and Kuhn, 2018)
- Optimization techniques (Namkoong and Duchi, 2016; Levy et al., 2020; Qi et al., 2021; Rafique et al., 2022)

Group DRO (Sagawa et al. 2020)

- Formulation: Minimax Risk Optimization

$$\min_{\mathbf{w}\in\mathcal{W}} \max_{i\in[m]} \left\{ R_i(\mathbf{w}) = \mathrm{E}_{\mathbf{z}\sim\mathcal{P}_i}\left[\ell(\mathbf{w};\mathbf{z})\right] \right\}$$

  - A finite number of m distributions

  - A new way for learning from multiple distributions

- Advantage: More Robust
  - A naïve approach

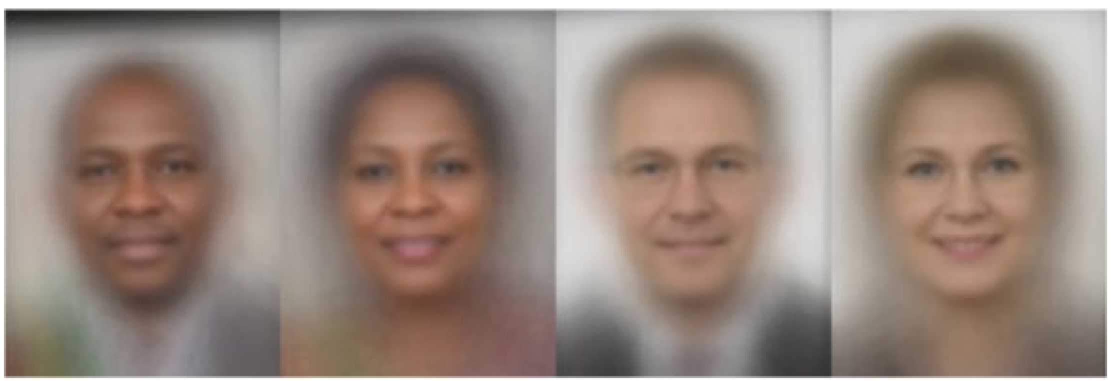
$$\min_{\mathbf{w}\in\mathcal{W}} \frac{1}{m}\sum_{i=1}^{m} R_i(\mathbf{w})$$

Application: Fairness

➢ Gender Classification (Buolamwini and Gebru 2018)

Gender Classifier	Darker Male	Darker Female	Lighter Male	Lighter Female	Largest Gap
Microsoft	94.0%	79.2%	100%	98.3%	20.8%
FACE++	99.3%	65.5%	99.2%	94.0%	33.8%
IBM	88.0%	65.3%	99.7%	92.9%	34.4%

High accuracy for lighter-skinned males, but worse accuracy for darker-skinned females



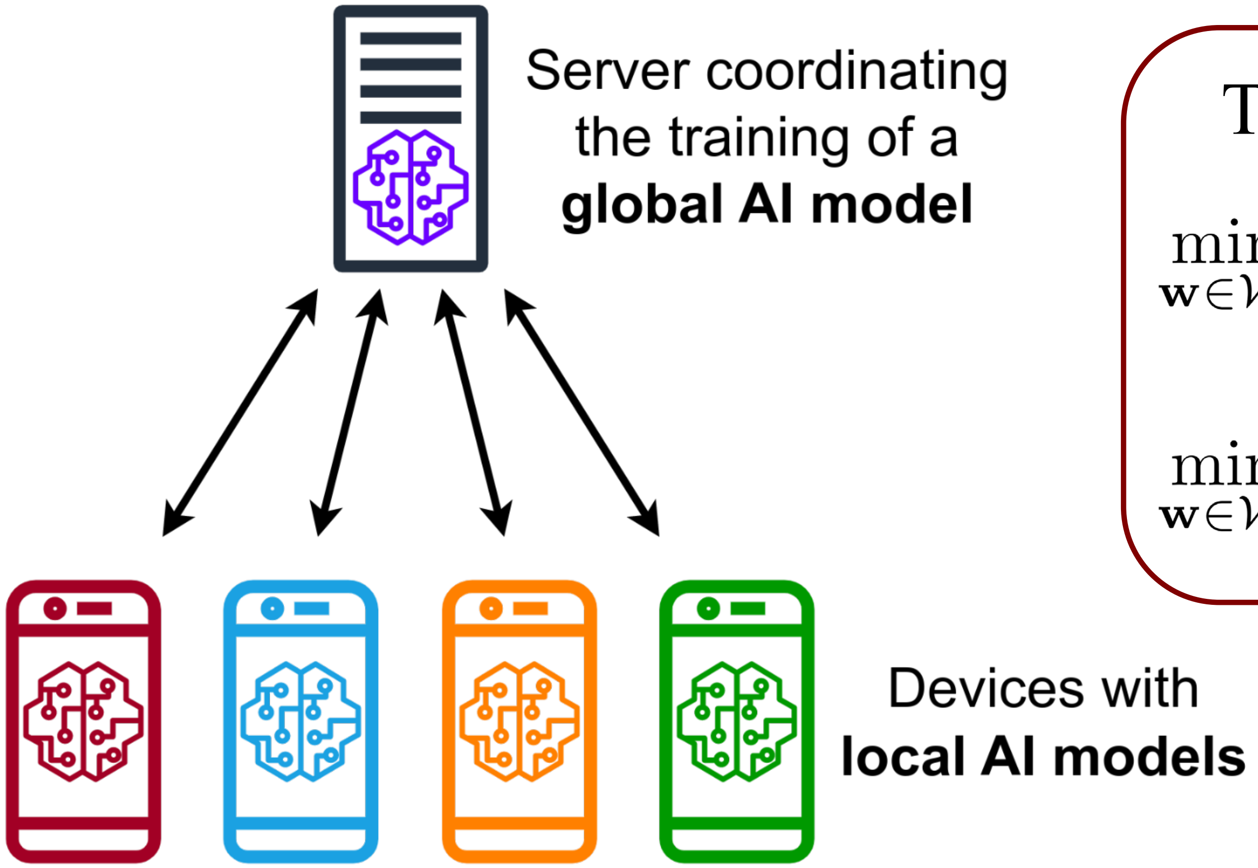
Solution ⇩

Optimizing performance across all groups

https://stanford-cs221.github.io/autumn2022-extra/modules/machine-learning/group-dro.pdf

Application: Federated Learning

- A Single Model facing Multiple Distributions



Two Choices

$$\min_{\mathbf{w}\in\mathcal{W}} \max_{i\in[m]} R_i(\mathbf{w})$$

$$\min_{\mathbf{w}\in\mathcal{W}} \frac{1}{m}\sum_{i=1}^{m} R_i(\mathbf{w})$$

https://en.wikipedia.org/wiki/Federated_learning

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Related Work I

➢ The Seminal Work of Sagawa et al. (ICLR 2020)

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- Introduce the problem of Group DRO
- Apply stochastic mirror descent (SMD)

for $t = 1, \ldots, T$ **do**
- $g \sim \text{Uniform}(1, \ldots, m)$ // Choose a group g at random
- $x, y \sim P_g$ // Sample x, y from group g
- $q' \leftarrow q^{(t-1)}$; $q'_g \leftarrow q'_g \exp(\eta_q \ell(\theta^{(t-1)}; (x, y)))$ // Update weights for group g
- $q^{(t)} \leftarrow q' / \sum_{g'} q'_{g'}$ // Renormalize q
- $\theta^{(t)} \leftarrow \theta^{(t-1)} - \eta_\theta q_g^{(t)} \nabla \ell(\theta^{(t-1)}; (x, y))$ // Use q to update θ

end

- A suboptimal $O(m^2 (\log m) / \epsilon^2)$ sample complexity

Related Work II

- The Work of Haghtalab et al. (NeurIPS 2022)

Nika Haghtalab, Michael I. Jordan, and Eric Zhao

University of California, Berkeley

- Try to improve the sample complexity by reusing samples

```
for a = 1, 2, ..., ⌈T/r⌉ do
  Realize ξ^{⊥(a)} at cost r;      // Sample datapoints from every distribution.
  for t = ar + 1 − r, ..., ar do
    Realize ξ^{q^(t)} at cost 1;   // Sample from adversary-selected distribution.
    Estimate gradients: ĝ_+^(t) = ĝ_+(ξ^{⊥(a)}, p^(t), q^(t)),  ĝ_-^(t) = ĝ_-(ξ^{q^(t)}, p^(t), q^(t));
    Run Hedge updates: p^(t+1) = Q_hedge(p^(t), ĝ_+^(t)), q^(t+1) = Q_hedge(q^(t), ĝ_+^(t));
  end for
end for
```

However, reusing samples introduces a dependence issue, making the analysis invalid.

Related Work III

- The Work of Soma et al. (2022)

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- Utilize online learning to reduce the sample complexity

for $t = 1, \ldots, T$ **do**
    Sample $i_t \sim q_t$.
    Call the stochastic oracle to obtain $z \sim P_{i_t}$.
    $\theta_{t+1} \leftarrow \mathrm{proj}_\Theta(\theta_t - \eta_{\theta,t} \nabla_\theta \ell(\theta_t; z))$
    $\nabla\Psi(\tilde{q}_{t+1}) \leftarrow \nabla\Psi(q_t) - \frac{\eta_q}{q_{t,i_t}} \ell(\theta_t; z) \mathbf{e}_{i_t}$; $q_{t+1} \leftarrow \mathrm{argmin}_{q \in Q} D_\Psi(q, \tilde{q}_{t+1})$, where $D_\Psi(x, y) = \Psi(x) - \Psi(y) - \nabla\Psi(x)^\top (y - x)$ is the Bregman divergence with respect to Ψ.
return $\frac{1}{T} \sum_{t=1}^{T} \theta_t$.

Online Convex Optimization

Multi-armed Bandits (MAB)

- Establish a nearly optimal $O(m(\log m)/\epsilon^2)$ complexity
- Suffer a dependence issue, but can be fixed

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Our Result I (Zhang et al. NeurIPS 2023)

- Minimax Risk Optimization

$$\min_{\mathbf{w}\in\mathcal{W}} \max_{i\in[m]} \left\{ R_i(\mathbf{w}) = \mathrm{E}_{\mathbf{z}\sim\mathcal{P}_i}\left[\ell(\mathbf{w};\mathbf{z})\right] \right\}$$

  - A finite number of m distributions

Equivalent

- Stochastic Convex-Concave Optimization

$$\min_{\mathbf{w}\in\mathcal{W}} \max_{\mathbf{q}\in\Delta_m} \left\{ \phi(\mathbf{w},\mathbf{q}) = \sum_{i=1}^{m} q_i R_i(\mathbf{w}) \right\}$$

  - $\Delta_m = \{\mathbf{q}\in\mathbb{R}^m : \mathbf{q}\geq 0, \sum_{i=1}^m q_i = 1\}$ is the $(m-1)$-dimensional simplex

  - Apply stochastic mirror descent (Nemirovski et al., 2009)

Performance Measure

- Stochastic Convex-Concave Optimization

$$\min_{\mathbf{w}\in\mathcal{W}} \max_{\mathbf{q}\in\Delta_m} \left\{ \phi(\mathbf{w},\mathbf{q}) = \sum_{i=1}^{m} q_i R_i(\mathbf{w}) \right\}$$

- Optimization Error of $(\bar{\mathbf{w}},\bar{\mathbf{q}})$

$$\epsilon_\phi(\bar{\mathbf{w}},\bar{\mathbf{q}}) = \max_{\mathbf{q}\in\Delta_m} \phi(\bar{\mathbf{w}},\mathbf{q}) - \min_{\mathbf{w}\in\mathcal{W}} \phi(\mathbf{w},\bar{\mathbf{q}})$$

  - Meaningful for Group DRO

$$\max_{i\in[m]} R_i(\bar{\mathbf{w}}) - \min_{\mathbf{w}\in\mathcal{W}} \max_{i\in[m]} R_i(\mathbf{w}) = \max_{\mathbf{q}\in\Delta_m} \sum_{i=1}^{m} q_i R_i(\bar{\mathbf{w}}) - \min_{\mathbf{w}\in\mathcal{W}} \max_{\mathbf{q}\in\Delta_m} \sum_{i=1}^{m} q_i R_i(\mathbf{w})$$

$$\leq \max_{\mathbf{q}\in\Delta_m} \sum_{i=1}^{m} q_i R_i(\bar{\mathbf{w}}) - \min_{\mathbf{w}\in\mathcal{W}} \sum_{i=1}^{m} \bar{q}_i R_i(\mathbf{w}) = \epsilon_\phi(\bar{\mathbf{w}},\bar{\mathbf{q}})$$

Stochastic Mirror Descent (SMD)

- Stochastic Convex-Concave Optimization

$$\min_{\mathbf{w}\in\mathcal{W}} \max_{\mathbf{q}\in\Delta_m} \left\{\phi(\mathbf{w},\mathbf{q}) = \sum_{i=1}^{m} q_i R_i(\mathbf{w})\right\}$$

  - Recall that $R_i(\mathbf{w}) = \mathrm{E}_{\mathbf{z}\sim\mathcal{P}_i}\left[\ell(\mathbf{w};\mathbf{z})\right]$
  - Stochastic Gradients at $(\mathbf{w}_t, \mathbf{q}_t)$

$$\mathbf{g}_w(\mathbf{w}_t,\mathbf{q}_t) = \sum_{i=1}^{m} q_{t,i}\nabla\ell(\mathbf{w}_t;\mathbf{z}_t^{(i)})$$

$$\mathbf{g}_q(\mathbf{w}_t,\mathbf{q}_t) = [\ell(\mathbf{w}_t;\mathbf{z}_t^{(1)}),\ldots,\ell(\mathbf{w}_t;\mathbf{z}_t^{(m)})]^\top$$

    - ✓ Draw m samples $\mathbf{z}_t^{(i)} \in \mathcal{P}_i,\ i=1,\ldots,m$

Stochastic Mirror Descent (SMD)

- Update by mirror descent

$$\mathbf{w}_{t+1} = \operatorname*{argmin}_{\mathbf{w}\in\mathcal{W}} \left\{\eta_w \langle \mathbf{g}_w(\mathbf{w}_t, \mathbf{q}_t), \mathbf{w} - \mathbf{w}_t\rangle + B_w(\mathbf{w}, \mathbf{w}_t)\right\}$$

$$\mathbf{q}_{t+1} = \operatorname*{argmin}_{\mathbf{q}\in\Delta_m} \left\{\eta_q \langle -\mathbf{g}_q(\mathbf{w}_t, \mathbf{q}_t), \mathbf{q} - \mathbf{q}_t\rangle + B_q(\mathbf{q}, \mathbf{q}_t)\right\}$$

✓ where $B_w(\mathbf{u}, \mathbf{v}) = \nu_w(\mathbf{u}) - \left[\nu_w(\mathbf{v}) + \langle \nabla \nu_w(\mathbf{v}), \mathbf{u} - \mathbf{v}\rangle\right]$

- Special cases:

$$\mathbf{w}_{t+1} = \Pi_{\mathcal{W}}\left[\mathbf{w}_t - \eta_w \mathbf{g}_w(\mathbf{w}_t, \mathbf{q}_t)\right]$$

$$q_{t+1,i} = \frac{q_{t,i} \exp\left(\eta_q \ell(\mathbf{w}_t; \mathbf{z}_t^{(i)})\right)}{\sum_{j=1}^m q_{t,j} \exp\left(\eta_q \ell(\mathbf{w}_t; \mathbf{z}_t^{(j)})\right)}, \ \forall i \in [m]$$

Theoretical Guarantee

Theorem 1 *By setting* $\eta_w = D^2\sqrt{\frac{8}{5T(D^2G^2+\ln m)}}$ *and* $\eta_q = (\ln m)\sqrt{\frac{8}{5T(D^2G^2+\ln m)}}$, *with probability at least* $1-\delta$,

$$\epsilon_\phi(\bar{\mathbf{w}}, \bar{\mathbf{q}}) \leq \left(8 + 2\ln\frac{2}{\delta}\right)\sqrt{\frac{10(D^2G^2+\ln m)}{T}} = O\left(\sqrt{\frac{\log m}{T}}\right)$$

- It requires m samples per iteration
- The total sample complexity is $O(m(\log m)/\epsilon^2)$
- Lower bound $\Omega(m/\epsilon^2)$ (Soma et al. 2022)

➢ Credit to Nemirovski et al. (2009, § 3.2)

3.2. Application to minimax stochastic problems. Consider the following minimax stochastic problem:

$$\min_{x\in X}\max_{1\leq i\leq m}\{f_i(x) = \mathbb{E}[F_i(x,\xi)]\}, \quad (3.18)$$

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Our Result II (Zhang et al. NeurIPS 2023)

Is it possible to reduce the number of samples per iteration from m to 1?

- The algorithm of Sagawa et al. (ICLR 2020)
  - Apply stochastic mirror descent with 1 sample pe iteration

$$\hat{\mathbf{g}}_w(\mathbf{w}_t, \mathbf{q}_t) = q_{t,i_t} m \nabla \ell(\mathbf{w}_t; \mathbf{z}_t^{(i_t)})$$

$$\hat{\mathbf{g}}_q(\mathbf{w}_t, \mathbf{q}_t) = [0, \ldots, m\ell(\mathbf{w}_t; \mathbf{z}_t^{(i_t)}), \ldots, 0]^\top$$

They are unbiased, but have very large variances.

  - Converge slowly, and have an $O(m^2 (\log m) / \epsilon^2)$ complexiy

Two-player Games

LAMDA
Learning And Mining from DatA
http://lamda.nju.edu.cn

- Stochastic Convex-Concave Optimization

$$\min_{\mathbf{w}\in\mathcal{W}} \max_{\mathbf{q}\in\Delta_m} \left\{ \phi(\mathbf{w}, \mathbf{q}) = \sum_{i=1}^{m} q_i R_i(\mathbf{w}) \right\}$$

- Two-player Games (Rakhlin and Sridharan, 2013)
  - The 1st player minimizes convex functions

$$\sum_{i=1}^{m} q_{1,i} R_i(\mathbf{w}), \ \sum_{i=1}^{m} q_{2,i} R_i(\mathbf{w}), \ \cdots, \ \sum_{i=1}^{m} q_{T,i} R_i(\mathbf{w})$$

  - The 2nd player maximizes linear functions

$$\sum_{i=1}^{m} q_i R_i(\mathbf{w}_1), \ \sum_{i=1}^{m} q_i R_i(\mathbf{w}_2), \ \cdots, \ \sum_{i=1}^{m} q_i R_i(\mathbf{w}_T)$$

Non-oblivious Online Learning

- The 1st player minimizes convex functions

$$\sum_{i=1}^{m} q_{1,i} R_i(\mathbf{w}), \ \sum_{i=1}^{m} q_{2,i} R_i(\mathbf{w}), \ \cdots, \ \sum_{i=1}^{m} q_{T,i} R_i(\mathbf{w})$$

  - Non-oblivious online convex optimization (OCO) with stochastic gradients

Stochastic gradients	We only have stochastic gradients of each online function $\sum_{i=1}^{m} q_{t,i} R_i(\cdot)$
Non-oblivious	The function $\sum_{i=1}^{m} q_{t,i} R_i(\cdot)$ depends on previous solutions $\mathbf{w}_1, \ldots, \mathbf{w}_{t-1}$

Non-oblivious Online Learning

- The 1st player minimizes convex functions

$$\sum_{i=1}^{m} q_{1,i} R_i(\mathbf{w}), \sum_{i=1}^{m} q_{2,i} R_i(\mathbf{w}), \cdots, \sum_{i=1}^{m} q_{T,i} R_i(\mathbf{w})$$

  - Non-oblivious online convex optimization (OCO) with stochastic gradients
  - Apply Stochastic Mirror Descent

$$\tilde{\mathbf{g}}_w(\mathbf{w}_t, \mathbf{q}_t) = \nabla \ell(\mathbf{w}_t; \mathbf{z}_t^{(i_t)})$$

→ Small variance

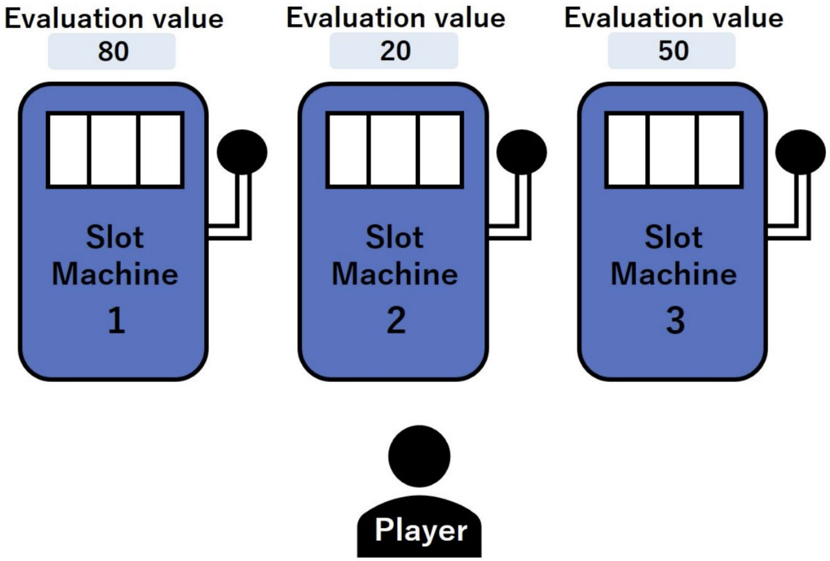
$$\mathbf{w}_{t+1} = \operatorname*{argmin}_{\mathbf{w} \in \mathcal{W}} \left\{ \eta_w \langle \tilde{\mathbf{g}}_w(\mathbf{w}_t, \mathbf{q}_t), \mathbf{w} - \mathbf{w}_t \rangle + B_w(\mathbf{w}, \mathbf{w}_t) \right\}$$

Non-oblivious Online Learning

- The 2nd player maximizes linear functions

$$\sum_{i=1}^{m} q_i R_i(\mathbf{w}_1),\ \sum_{i=1}^{m} q_i R_i(\mathbf{w}_2),\ \cdots,\ \sum_{i=1}^{m} q_i R_i(\mathbf{w}_T)$$

  - Non-oblivious multi-armed bandits (MAB) with stochastic rewards



http://www.apsipa.org/proceedings/2021/pdfs/0001899.pdf

Non-oblivious Online Learning

- The 2nd player maximizes linear functions

$$\sum_{i=1}^{m} q_i R_i(\mathbf{w}_1), \sum_{i=1}^{m} q_i R_i(\mathbf{w}_2), \cdots, \sum_{i=1}^{m} q_i R_i(\mathbf{w}_T)$$

- Non-oblivious multi-armed bandits (MAB) with stochastic rewards
- Apply Exp3-IX for non-oblivious MAB (Neu, 2015)

$$\tilde{s}_{t,i} = \boxed{\frac{1 - \ell(\mathbf{w}_t, \mathbf{z}_t^{(i_t)})}{q_{t,i} + \gamma} \cdot \mathbb{I}[i_t = i]}$$

→ Bias-Variance tradeoff

$$\mathbf{q}_{t+1} = \operatorname*{argmin}_{\mathbf{q} \in \Delta_m} \left\{ \eta_q \langle \tilde{\mathbf{s}}_t, \mathbf{q} - \mathbf{q}_t \rangle + B_q(\mathbf{q}, \mathbf{q}_t) \right\}$$

Theoretical Guarantee

Theorem 2 *By setting* $\eta_w = \frac{2D}{G\sqrt{5T}}$, $\eta_q = \sqrt{\frac{\ln m}{mT}}$ *and* $\gamma = \frac{\eta_q}{2}$, *with probability at least* $1-\delta$,

$$\begin{aligned}\epsilon_\phi(\bar{\mathbf{w}}, \bar{\mathbf{q}}) \leq & DG\sqrt{\frac{1}{T}}\left(2\sqrt{5} + 8\sqrt{\frac{1}{2}\ln\frac{2}{\delta}}\right) + 3\sqrt{\frac{m\ln m}{T}} + \sqrt{\frac{1}{2T}} \\ & + \left(\sqrt{\frac{m}{T\ln m}} + \sqrt{\frac{1}{2T}} + \frac{1}{T}\right)\ln\frac{6}{\delta} \\ = & O\left(\sqrt{\frac{m\log m}{T}}\right)\end{aligned}$$

- It requires 1 samples per iteration
- The total sample complexity is $O(m(\log m)/\epsilon^2)$
- Lower bound $\Omega(m/\epsilon^2)$ (Soma et al. 2022)

Experiments: Convergence Rate

- Adult dataset, Logistic loss, 6 Groups

- SMD(1), $O\left(m\sqrt{\frac{\log m}{T}}\right)$

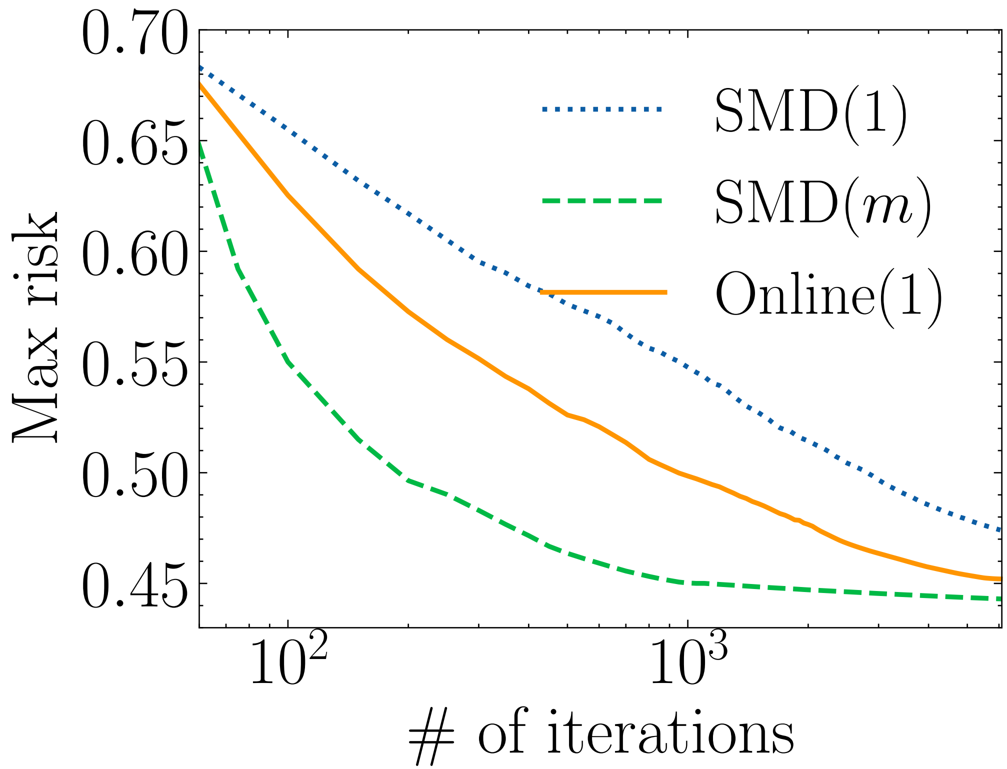
(Sagawa et al. ICLR 2020)

- SMD(m), $O\left(\sqrt{\frac{\log m}{T}}\right)$

Our Alg. 1

- Online(1), $O\left(\sqrt{\frac{m(\log m)}{T}}\right)$

Our Alg. 2



Experiments: Sample Complexity

Learning And Mining from DatA
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- Adult dataset, Logistic loss, 6 Groups

■ SMD(1), $O\left(\frac{m^2(\log m)}{\epsilon^2}\right)$

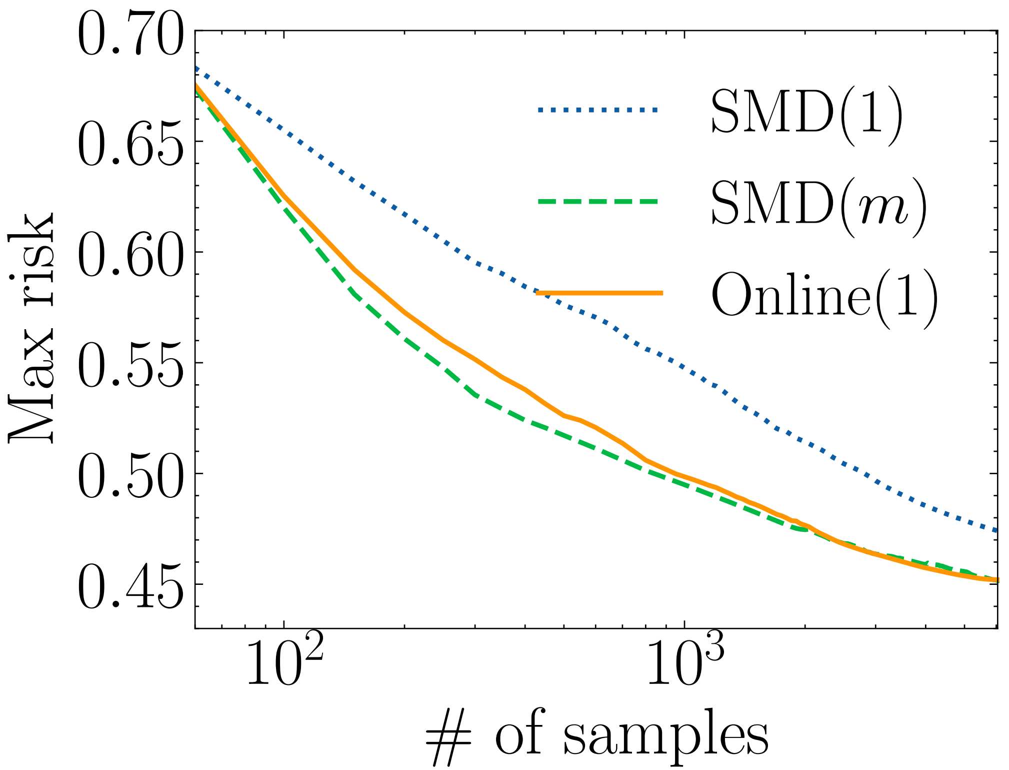
(Sagawa et al. ICLR 2020)

■ SMD(m), $O\left(\frac{m(\log m)}{\epsilon^2}\right)$

Our Alg. 1

■ Online(1), $O\left(\frac{m(\log m)}{\epsilon^2}\right)$

Our Alg. 2



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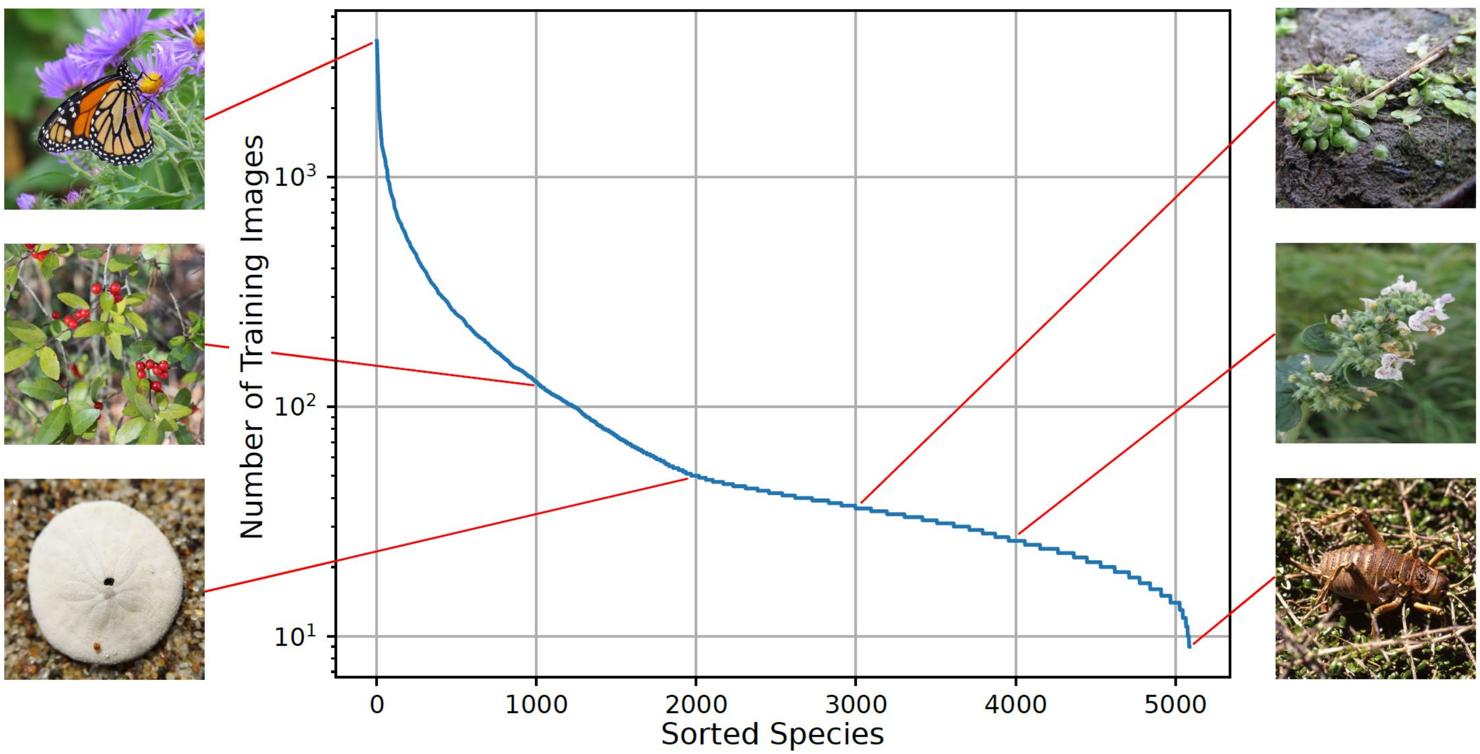
Imbalanced datasets

- iNaturalist dataset, consisting of 859,000 images from over 5,000 different species (Horn et al, 2018)



Distribution of training images per species

Stochastic Mirror Descent (SMD)

- Stochastic Convex-Concave Optimization

$$\min_{\mathbf{w}\in\mathcal{W}} \max_{\mathbf{q}\in\Delta_m} \left\{ \phi(\mathbf{w}, \mathbf{q}) = \sum_{i=1}^{m} q_i R_i(\mathbf{w}) \right\}$$

- Recall that $R_i(\mathbf{w}) = \mathrm{E}_{\mathbf{z}\sim\mathcal{P}_i}\left[\ell(\mathbf{w};\mathbf{z})\right]$
- Stochastic Gradients at $(\mathbf{w}_t, \mathbf{q}_t)$

$$\mathbf{g}_w(\mathbf{w}_t, \mathbf{q}_t) = \sum_{i=1}^{m} q_{t,i} \nabla \ell(\mathbf{w}_t; \mathbf{z}_t^{(i)})$$

$$\mathbf{g}_q(\mathbf{w}_t, \mathbf{q}_t) = [\ell(\mathbf{w}_t; \mathbf{z}_t^{(1)}), \ldots, \ell(\mathbf{w}_t; \mathbf{z}_t^{(m)})]^\top$$

✓ Draw *m* samples $\mathbf{z}_t^{(i)} \in \mathcal{P}_i,\ i = 1, \ldots, m$

It draws **the same number** of samples from every distribution.

GDRO Under Imbalanced Setting

- n_i be the number of samples can be drawn from $\mathcal{P}_i$

$$n_1 \geq n_2 \geq \cdots \geq n_m$$

- A naive **baseline**: running SMD for n_m rounds

n_m rounds

$$\mathbf{g}_w(\mathbf{w}_t, \mathbf{q}_t) = \sum_{i=1}^{m} q_{t,i} \nabla \ell(\mathbf{w}_t; \mathbf{z}_t^{(i)})$$

$$\mathbf{g}_q(\mathbf{w}_t, \mathbf{q}_t) = [\ell(\mathbf{w}_t; \mathbf{z}_t^{(1)}), \ldots, \ell(\mathbf{w}_t; \mathbf{z}_t^{(m)})]^\top$$

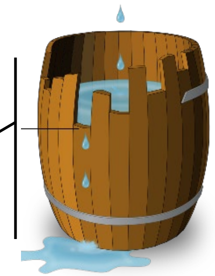
$$\mathbf{w}_{t+1} = \operatorname*{argmin}_{\mathbf{w} \in \mathcal{W}} \left\{ \eta_w \langle \mathbf{g}_w(\mathbf{w}_t, \mathbf{q}_t), \mathbf{w} - \mathbf{w}_t \rangle + B_w(\mathbf{w}, \mathbf{w}_t) \right\}$$

$$\mathbf{q}_{t+1} = \operatorname*{argmin}_{\mathbf{q} \in \Delta_m} \left\{ \eta_q \langle -\mathbf{g}_q(\mathbf{w}_t, \mathbf{q}_t), \mathbf{q} - \mathbf{q}_t \rangle + B_q(\mathbf{q}, \mathbf{q}_t) \right\}$$

Limitations of Baseline

1. The optimization error is determined by n_m
   - According to Theorem 1, we have

$$\epsilon_\phi(\bar{\mathbf{w}}, \bar{\mathbf{q}}) = O\left(\sqrt{\frac{\log m}{n_m}}\right)$$



Barrel Effect

2. A large amount of samples are wasted
   - For distribution $\mathcal{P}_1$, $n_1 - n_m$ samples are wasted
   - For distribution $\mathcal{P}_2$, $n_2 - n_m$ samples are wasted

   ……

   - For distribution $\mathcal{P}_{m-1}$, $n_{m-1} - n_m$ samples are wasted

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Our Result III (Zhang et al. NeurIPS 2023)

- Applying Non-uniform Sampling
  - Run n_1 iterations, and draw a sample from $\mathcal{P}_i$ with probability $p_i = n_i / n_1$

$$\text{Expected \# of Samples: } n_1 \cdot \frac{n_i}{n_1} = n_i$$

- Updating according to SMD
  - Construct stochastic gradients

$$\mathbf{g}_w(\mathbf{w}_t, \mathbf{q}_t) = \sum_{i \in C_t} \frac{q_{t,i}}{p_i} \nabla \ell(\mathbf{w}_t; \mathbf{z}_t^{(i)})$$

$$[\mathbf{g}_q(\mathbf{w}_t, \mathbf{q}_t)]_i = \begin{cases} \ell(\mathbf{w}_t; \mathbf{z}_t^{(i)}) / p_i, & i \in C_t \\ 0, & \text{otherwise} \end{cases}$$

It yields very slow convergence due to the large variance caused by $1/p_m = n_1/n_m$.

✓ C_t is the set of indexes of selected distributions

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✓ C_t is the set of indexes of selected distributions

Our Result III (Zhang et al. NeurIPS 2023)

- A Weighted GDRO Problem

$$\min_{\mathbf{w}\in\mathcal{W}} \max_{\mathbf{q}\in\Delta_m} \left\{ \varphi(\mathbf{w},\mathbf{q}) = \sum_{i=1}^{m} q_i \cdot p_i R_i(\mathbf{w}) \right\}$$

  - The more the number of samples, the larger the weights

- Updating according to SMD
  - Construct stochastic gradients

$$\mathbf{g}_w(\mathbf{w}_t,\mathbf{q}_t) = \sum_{i\in C_t} q_{t,i} \nabla \ell(\mathbf{w}_t;\mathbf{z}_t^{(i)})$$

$$[\mathbf{g}_q(\mathbf{w}_t,\mathbf{q}_t)]_i = \begin{cases} \ell(\mathbf{w}_t;\mathbf{z}_t^{(i)}), & i \in C_t \\ 0, & \text{otherwise} \end{cases}$$

    - ✓ C_t is the set of indexes of selected distributions

Advantages of Weighted GDRO

- A Weighted GDRO Problem

$$\min_{\mathbf{w}\in\mathcal{W}} \max_{\mathbf{q}\in\Delta_m} \left\{ \varphi(\mathbf{w},\mathbf{q}) = \sum_{i=1}^{m} q_i \cdot p_i R_i(\mathbf{w}) \right\}$$

  - Optimization Error of $(\bar{\mathbf{w}},\bar{\mathbf{q}})$

$$\max_{i\in[m]} p_i R_i(\bar{\mathbf{w}}) - \min_{\mathbf{w}\in\mathcal{W}} \max_{\mathbf{q}\in\Delta_m} \varphi(\mathbf{w},\mathbf{q}) \le \epsilon_\varphi(\bar{\mathbf{w}},\bar{\mathbf{q}})$$

- Risk of Each Distribution

$$R_i(\bar{\mathbf{w}}) \le \frac{1}{p_i} \min_{\mathbf{w}\in\mathcal{W}} \max_{\mathbf{q}\in\Delta_m} \varphi(\mathbf{w},\mathbf{q}) + \frac{1}{p_i}\epsilon_\varphi(\bar{\mathbf{w}},\bar{\mathbf{q}})$$

More Samples → Larger Weights → Faster Rates

Theoretical Guarantee

Theorem 3 *By setting* $\eta_w = D^2\sqrt{\frac{8}{5n_1(D^2G^2+\ln m)}}$ *and* $\eta_q = (\ln m)\sqrt{\frac{8}{5n_1(D^2G^2+\ln m)}}$, *with probability at least* $1-\delta$,

$$R_i(\bar{\mathbf{w}}) - \frac{1}{p_i}p_\varphi^* \leq \mu(\delta)\frac{\sqrt{10(D^2G^2+\ln m)n_1}}{n_i}$$

$$= O\left(\frac{\sqrt{n_1 \log m}}{n_i}\right)$$

Distribution-dependent

- The $O\left(\frac{\sqrt{n_1\log m}}{n_i}\right)$ rate is better than Baseline's $O\left(\sqrt{\frac{\log m}{n_m}}\right)$ rate when $n_i \geq \sqrt{n_1 n_m}$
- For distributions $\mathcal{P}_1$, the rate $O\left(\sqrt{\frac{\log m}{n_1}}\right)$ is nearly optimal

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Our Result IV (Zhang et al. NeurIPS 2023)

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- Applying Mini-batches
  - Run n_m iterations, and draw n_i/n_m sample from $\mathcal{P}_i$

$$\text{\# of Samples: } n_m \cdot \frac{n_i}{n_m} = n_i$$

- Stochastic Gradients with Elements Having Small Variance

$$\mathbf{g}_w(\mathbf{w}_t, \mathbf{q}_t) = \sum_{i=1}^{m} q_{t,i} \left(\frac{n_m}{n_i} \sum_{j=1}^{n_i/n_m} \nabla \ell(\mathbf{w}_t; \mathbf{z}_t^{(i,j)}) \right)$$

$$\mathbf{g}_q(\mathbf{w}_t, \mathbf{q}_t) = \left[\frac{n_m}{n_1} \sum_{j=1}^{n_1/n_m} \ell(\mathbf{w}_t; \mathbf{z}_t^{(1,j)}), \ldots, p_m \ell(\mathbf{w}_t; \mathbf{z}_t^{(m)}) \right]^\top$$

Two Challenges

1. The performance of SMD does not depend on variance

➢ Stochastic Mirror-Prox Algorithm (SMPA) (Juditsky et al., 2011)

- Basically, it performs SMD twice in each iteration
- The convergence rate depends on the variance

2. The whole gradient still have a large variance

➢ A Weighted GDRO Problem

$$\min_{\mathbf{w}\in\mathcal{W}} \max_{\mathbf{q}\in\Delta_m} \left\{ \varphi(\mathbf{w}, \mathbf{q}) = \sum_{i=1}^{m} q_i \cdot p_i R_i(\mathbf{w}) \right\}$$

- Set larger weights for distributions with smaller variance

Advantages of Weighted GDRO

- A Weighted GDRO Problem

$$\min_{\mathbf{w}\in\mathcal{W}} \max_{\mathbf{q}\in\Delta_m} \left\{ \varphi(\mathbf{w},\mathbf{q}) = \sum_{i=1}^{m} q_i \cdot p_i R_i(\mathbf{w}) \right\}$$

- Optimization Error of $(\bar{\mathbf{w}}, \bar{\mathbf{q}})$

$$\max_{i\in[m]} p_i R_i(\bar{\mathbf{w}}) - \min_{\mathbf{w}\in\mathcal{W}} \max_{\mathbf{q}\in\Delta_m} \varphi(\mathbf{w},\mathbf{q}) \leq \epsilon_\varphi(\bar{\mathbf{w}}, \bar{\mathbf{q}})$$

- Risk of Each Distribution

$$R_i(\bar{\mathbf{w}}) \leq \frac{1}{p_i} \min_{\mathbf{w}\in\mathcal{W}} \max_{\mathbf{q}\in\Delta_m} \varphi(\mathbf{w},\mathbf{q}) + \frac{1}{p_i}\epsilon_\varphi(\bar{\mathbf{w}}, \bar{\mathbf{q}})$$

More Samples → Smaller Variances → Larger Weights → Faster Rates

Theoretical Guarantee

Theorem 4 *By setting* $p_i = \frac{1/\sqrt{n_m}+1}{1/\sqrt{n_m}+\sqrt{n_m/n_i}}$, *with high probability*

$$R_i(\bar{\mathbf{w}}) - \frac{1}{p_i} p_\varphi^* = O\left(\left(\frac{1}{n_m} + \frac{1}{\sqrt{n_i}}\right)\sqrt{\kappa + \ln^2 m}\right)$$

Distribution -dependent

- ◻ A fast $O\left(\frac{\log m}{\sqrt{n_i}}\right)$ rate for distributions $\mathcal{P}_i$ such that $n_i \leq n_m^2$
- ◻ In contrast, the rate of Baseline is $O\left(\sqrt{\frac{\log m}{n_m}}\right)$
- ◻ A fast $O\left(\frac{\log m}{n_m}\right)$ rate for distributions $\mathcal{P}_i$ such that $n_i \geq n_m^2$
  - ✓ There exists a performance limit

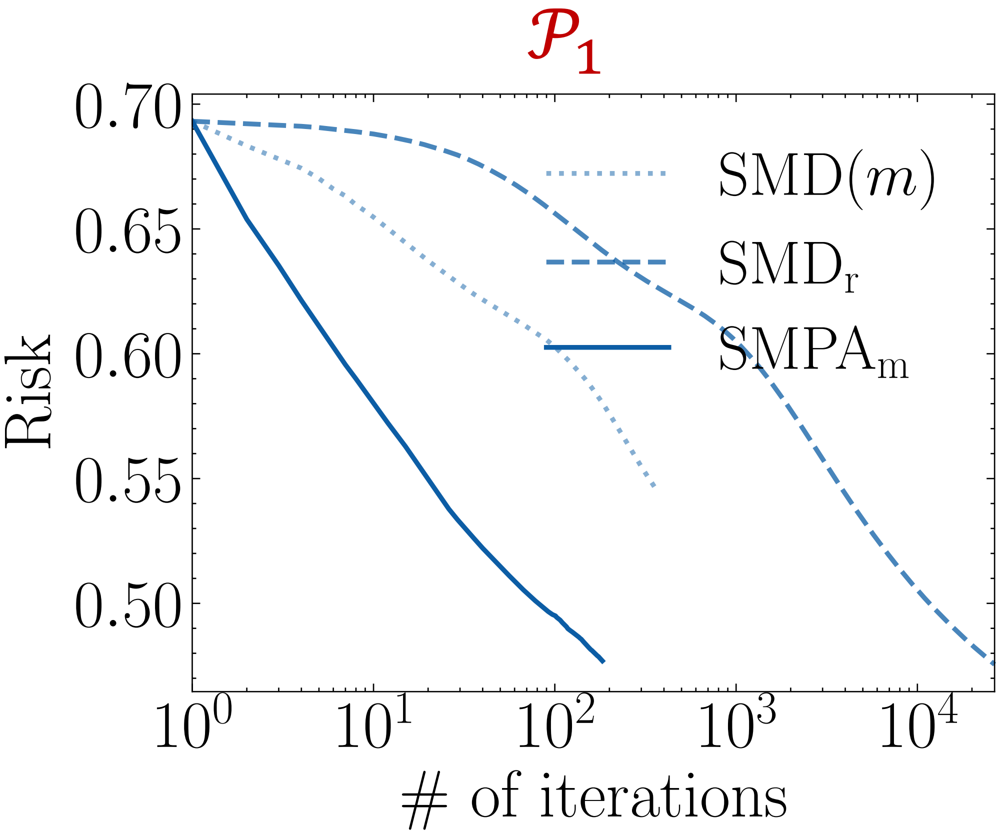
Experiments: Convergence Rate

- Adult dataset, Logistic loss, 6 Groups
- # of Samples: 26656, 11518, 1780, 1720, 998, and 364

■ SMD(m), $O\left(\sqrt{\frac{\log m}{n_6}}\right)$
Our Alg. 1

■ SMD_r, $O\left(\sqrt{\frac{\log m}{n_1}}\right)$ ✓
Our Alg. 3

■ SMPA_m, $O\left(\frac{\log m}{\sqrt{n_1}}\right)$ ✓
Our Alg. 4



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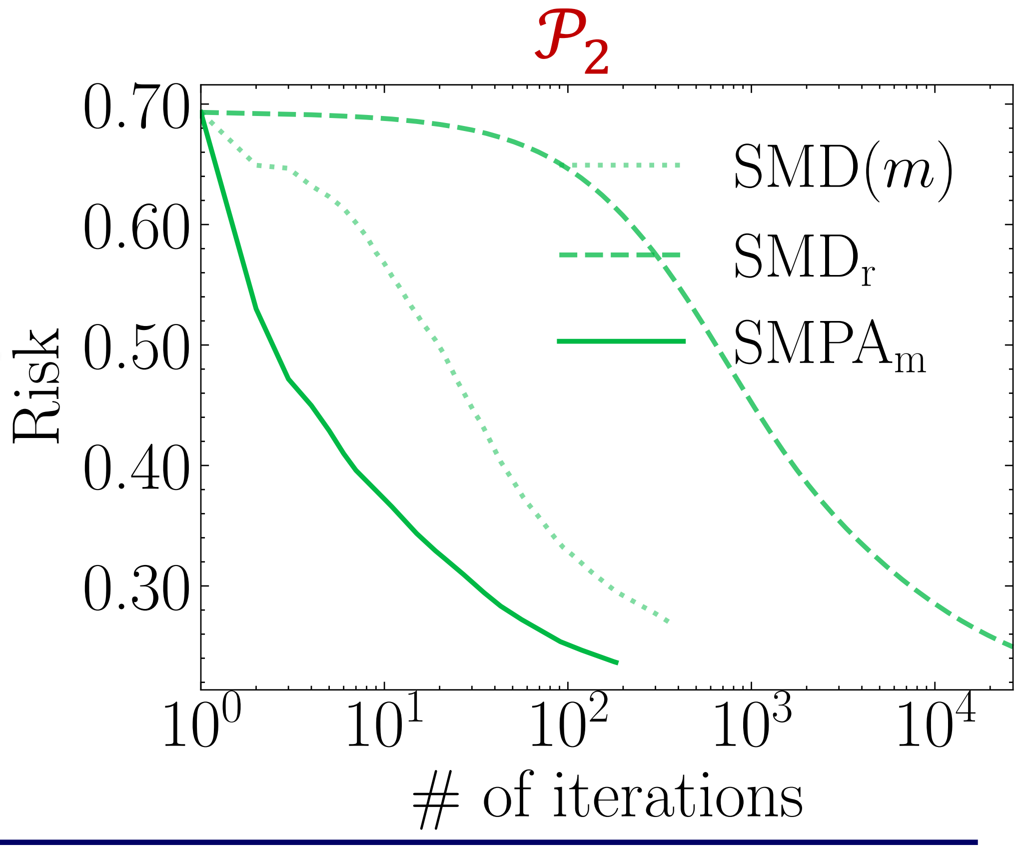

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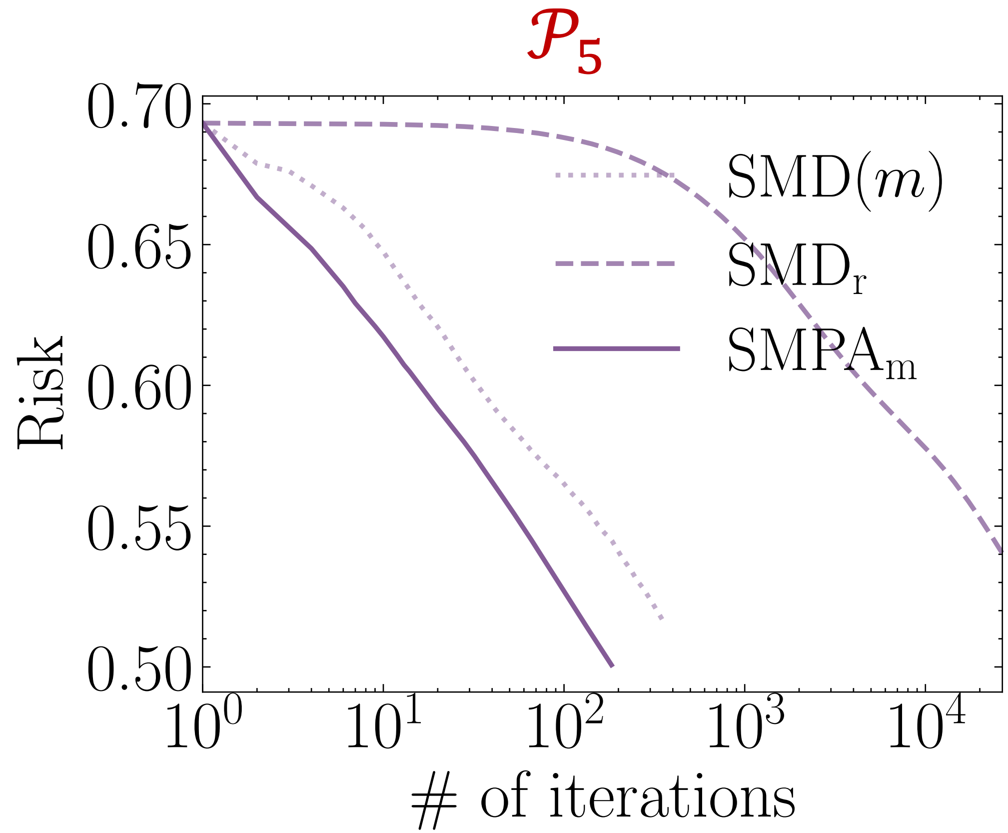
■ SMPA_m, $O\left(\frac{\log m}{\sqrt{n_5}}\right)$

Our Alg. 4



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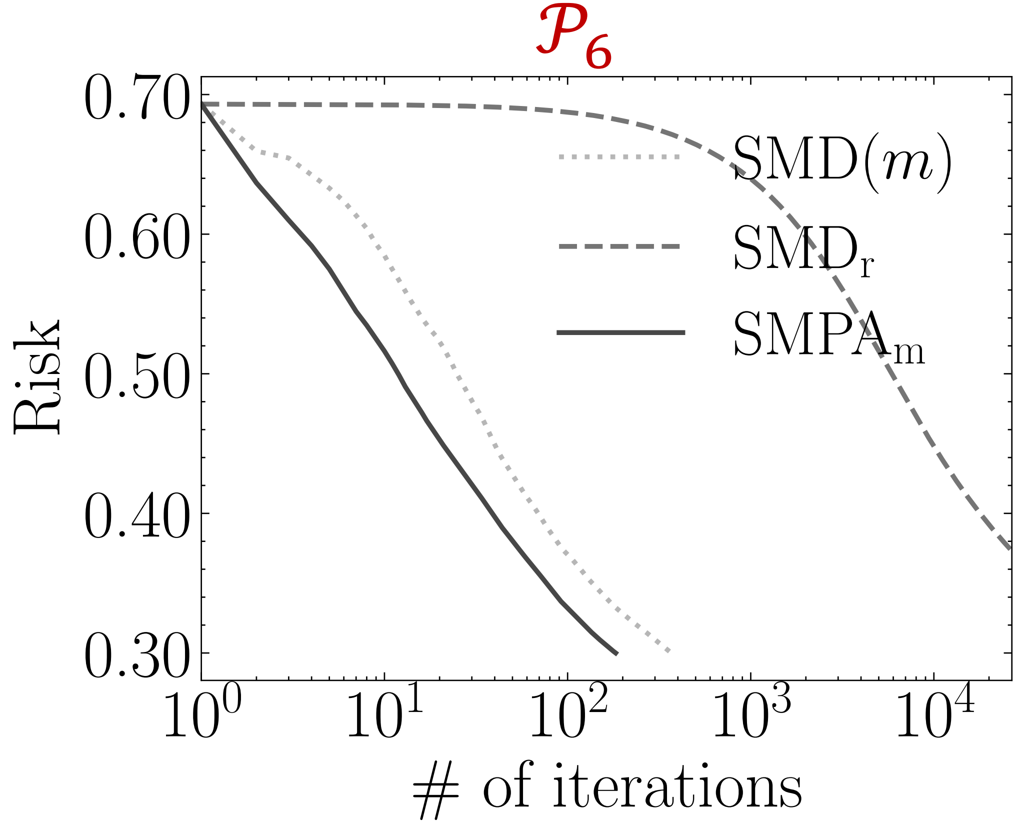

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Conclusion

- GDRO——Minimax Risk Optimization

$$\min_{\mathbf{w}\in\mathcal{W}} \max_{i\in[m]} \left\{R_i(\mathbf{w}) = \mathrm{E}_{\mathbf{z}\sim\mathcal{P}_i}\left[\ell(\mathbf{w};\mathbf{z})\right]\right\}$$

1. Stochastic Mirror Descent, $O(m(\log m)/\epsilon^2)$
2. Non-oblivious Online Learning, $O(m(\log m)/\epsilon^2)$

- GDRO with Imbalanced Data

$$\min_{\mathbf{w}\in\mathcal{W}} \max_{\mathbf{q}\in\Delta_m} \left\{\varphi(\mathbf{w},\mathbf{q}) = \sum_{i=1}^{m} q_i \cdot p_i R_i(\mathbf{w})\right\}$$

1. Stochastic Mirror Descent with Non-uniform Sampling
2. Stochastic Mirror-Prox Algorithm with Mini-batches

- Distribution-dependent Convergence Rates

Future Work

- More Investigations of the Imbalanced Scenario
  - Understand the red terms below

$$R_i(\bar{\mathbf{w}}) - \frac{1}{p_i} p_\varphi^* = O(\cdot)$$

- Minimax Excess Risk Optimization (MERO) (Agarwal and Zhang, 2022)

$$\min_{\mathbf{w}\in\mathcal{W}} \max_{i\in[m]} \left\{ \mathrm{E}_{\mathbf{z}\sim\mathcal{P}_i}\left[\ell(\mathbf{w};\mathbf{z})\right] - \min_{\mathbf{w}\in\mathcal{W}} \mathrm{E}_{\mathbf{z}\sim\mathcal{P}_i}\left[\ell(\mathbf{w};\mathbf{z})\right] \right\}$$

  - Subtracting the intrinsic difficulty of each distribution
  - Efficient stochastic algorithms (Zhang et al. 2023)

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